

EXTREME VALUES OF ZETA AND L -FUNCTIONS

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1. INTRODUCTION

In this paper we introduce a “resonance” method to produce large values of $|\zeta(\frac{1}{2} + it)|$ and large and small central values of L -functions.

Theorem 1. *If T is sufficiently large then there exists $t \in [T, 2T]$ such that*

$$|\zeta(\tfrac{1}{2} + it)| \geq \exp\left((1 + o(1)) \frac{\sqrt{\log T}}{\sqrt{\log \log T}}\right).$$

Moreover uniformly in the range $3 \leq V \leq \frac{1}{5} \sqrt{\log T / \log \log T}$ we have that

$$\text{meas}\{t \in [T, 2T] : |\zeta(\tfrac{1}{2} + it)| \geq e^V\} \gg \frac{T}{(\log T)^4} \exp\left(-10 \frac{V^2}{\log \frac{\log T}{8V^2 \log V}}\right).$$

The problem of obtaining large values of $|\zeta(\frac{1}{2} + it)|$ was first considered by E.C. Titchmarsh who showed that there exist arbitrarily large t with $|\zeta(\frac{1}{2} + it)| \geq \exp(\log^\alpha t)$ for any $\alpha < \frac{1}{2}$ (see Theorem 8.12 of [15]). In [9] H.L. Montgomery proved that, assuming the Riemann Hypothesis, there exist arbitrarily large values t such that

$$|\zeta(\tfrac{1}{2} + it)| \gg \exp\left(\frac{1}{20} \frac{\sqrt{\log |t|}}{\sqrt{\log \log |t|}}\right).$$

R. Balasubramanian and K. Ramachandra [2] proved a similar result unconditionally, showing that there are arbitrarily large t such that

$$|\zeta(\tfrac{1}{2} + it)| \gg \exp\left(B \frac{\sqrt{\log |t|}}{\sqrt{\log \log |t|}}\right),$$

for some positive constant B . Their method is based on obtaining lower bounds for the moments $\int_T^{2T} |\zeta(\frac{1}{2} + it)|^{2k} dt$. Later Balasubramanian [1] optimized their argument and found that $B = 0.530 \dots$ is permissible.¹

The author is partially supported by the National Science Foundation (DMS 0500711) and the American Institute of Mathematics (AIM).

¹The value of B stated by him is $B = 0.75 \dots$, but there appears to be a numerical error in the calculation.

As well as improving these results, our Theorem above suggests that there should be still larger values of $|\zeta(\frac{1}{2} + it)|$. A. Selberg (see [15]) has shown that as t varies between T and $2T$, $\log |\zeta(\frac{1}{2} + it)|$ has an approximately Gaussian distribution with mean 0 and variance $\sim \frac{1}{2} \log \log T$. This suggests that the set of $t \in [T, 2T]$ with $|\zeta(\frac{1}{2} + it)| \geq e^V$ should have measure about $T \exp(-V^2 / \log \log T)$. Our Theorem furnishes a lower bound for this measure of the type $T \exp(-cV^2 / \log \log T)$ for some positive constant c uniformly in the range $\log \log T \leq V \leq (\log T)^{\frac{1}{2} - \delta}$ for any fixed $\delta > 0$. If this type of estimate were to persist for larger V , then we would expect to find values of $|\zeta(\frac{1}{2} + it)|$ of size $\exp(C\sqrt{\log T \log \log T})$ for some positive constant C . Indeed, recently D.W. Farmer, S.M. Gonek and C.P. Hughes [4] have suggested, based on several interesting heuristic considerations, that the maximum size of $|\zeta(\frac{1}{2} + it)|$ is about $\exp(C\sqrt{\log t \log \log t})$ with $C = 1/\sqrt{2} + o(1)$.

Complementing the lower bound of Theorem 1, we have shown in [14] that assuming the Riemann hypothesis

$$\text{meas}\{t \in [T, 2T] : |\zeta(\frac{1}{2} + it)| \geq e^V\} \ll T \exp\left(- (1 + o(1)) \frac{V^2}{\log \log T}\right),$$

in the range $10\sqrt{\log \log T} \leq V = o(\log \log T \log \log \log T)$. When $V \geq \log \log T \log \log \log T$ this measure is $\ll T \exp(-cV \log V)$ for some positive constant c . For a precise statement see the Theorem in [14].

The main idea of our proof is to find a Dirichlet polynomial $R(t) = \sum_{n \leq N} r(n)n^{-it}$ which ‘resonates’ with $\zeta(\frac{1}{2} + it)$ and picks out its large values. Precisely, we will compute the smoothed moments

$$(1) \quad M_1(R, T) = \int_{-\infty}^{\infty} |R(t)|^2 \Phi\left(\frac{t}{T}\right) dt, \quad \text{and} \quad M_2(R, T) = \int_{-\infty}^{\infty} \zeta\left(\frac{1}{2} + it\right) |R(t)|^2 \Phi\left(\frac{t}{T}\right) dt.$$

Here Φ denotes a smooth, non-negative function, compactly supported in $[1, 2]$, with $\Phi(y) \leq 1$ for all y , and $\Phi(y) = 1$ for $5/4 \leq y \leq 7/4$. Plainly

$$\max_{T \leq t \leq 2T} |\zeta(\frac{1}{2} + it)| \geq \frac{|M_2(R, T)|}{M_1(R, T)}.$$

When $N \leq T^{1-\epsilon}$ we may evaluate $M_1(R, T)$ and $M_2(R, T)$ easily. These are two quadratic forms in the unknown coefficients $r(n)$, and the problem thus reduces to maximizing the ratio of these quadratic forms. Solving this optimization problem we obtain Theorem 1.

This method generalizes readily to provide large and small central values in families of L -functions. By contrast, the method of Montgomery does not appear to generalize to this situation. Recently Z. Rudnick and the author ([11] and [12]) found a flexible method to obtain lower bounds for moments in many families of L -functions, but the bounds obtained here are superior.

Theorem 2. *Let X be large. There exists a fundamental discriminant d with $X \leq |d| \leq 2X$ such that*

$$L\left(\frac{1}{2}, \chi_d\right) \geq \exp\left(\left(\frac{1}{\sqrt{5}} + o(1)\right) \frac{\sqrt{\log X}}{\log \log X}\right).$$

Moreover, there exists a fundamental discriminant d with $X \leq |d| \leq 2X$ such that

$$|L(\tfrac{1}{2}, \chi_d)| \leq \exp \left(- \left(\frac{1}{\sqrt{5}} + o(1) \right) \frac{\sqrt{\log X}}{\log \log X} \right).$$

Here χ_d denotes the real primitive character associated to the fundamental discriminant d .

Previously, D.R. Heath-Brown (unpublished, see [6]) had shown that there are arbitrarily large fundamental discriminants d such that

$$L(\tfrac{1}{2}, \chi_d) \gg \exp \left(C \frac{\sqrt{\log |d|}}{\log \log |d|} \right),$$

for some positive constant C . Heath-Brown's idea was extended by J. Hoffstein and P. Lockhart [6] to prove a similar result for quadratic twists of any modular form. Our method may be adapted to give an analogous improvement of their result.

S.D. Chowla has conjectured that $L(\tfrac{1}{2}, \chi_d) > 0$ for all fundamental discriminants d . From [13] we know that $L(\tfrac{1}{2}, \chi_d) \neq 0$ for a large proportion ($\frac{7}{8}$) of fundamental discriminants d , and from [3] that $L(\tfrac{1}{2}, \chi_d) > 0$ for a positive proportion of fundamental discriminants d . Nevertheless, Theorem 2 tells us that there are very small values of $L(\tfrac{1}{2}, \chi_d)$, and arguing as in Theorem 1 we can also show that there are $\gg X \exp(-C \log X / \log \log X)$ discriminants d with such a small value of $L(\tfrac{1}{2}, \chi_d)$.

We give one more example of this method. Let k denote an even integer and let $H_k = H_k(1)$ denote the set of Hecke eigencuspforms of weight k for the full modular group $\Gamma = SL_2(\mathbb{Z})$. We write the Fourier expansion of $f \in H_k$ as

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz)$$

and normalize so that $\lambda_f(1) = 1$. Note that, with our normalization, Deligne's bound reads $|\lambda_f(n)| \leq d(n)$ although we do not require it here. Associated to f is the L -function

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}.$$

Recall that the sign of the functional equation for $L(s, f)$ is i^k . When $k \equiv 2 \pmod{4}$ it follows that the central values $L(\tfrac{1}{2}, f)$ equal zero.

Theorem 3. *For large $k \equiv 0 \pmod{4}$ there exists $f \in H_k$ with*

$$L(\tfrac{1}{2}, f) \geq \exp \left((1 + o(1)) \frac{\sqrt{2 \log k}}{\sqrt{\log \log k}} \right).$$

There also exists $f \in H_k$ with

$$L(\tfrac{1}{2}, f) \leq \exp \left(- (1 + o(1)) \frac{\sqrt{2 \log k}}{\sqrt{\log \log k}} \right).$$

In Theorem 1 we have attempted to optimize the large values of $|\zeta(\frac{1}{2} + it)|$ produced by our method. In Theorems 2 and 3 we have tried instead to keep the exposition simple, and not pushed the method to its limit. For example, with greater work we could take a longer resonator, allowing us to replace the $1/\sqrt{5}$ appearing in Theorem 2 with $\sqrt{3}$.

The resonance method is useful in producing omega results in other contexts as well. For example, in work in progress A. Booker and the author have used it to obtain large character sums improving and simplifying the results in [5]. Using this method and adding their ideas, N. Ng [10] has obtained large and small values of $|\zeta'(\rho)|$ where ρ runs over zeros of $\zeta(s)$, and D. Milicevic [8] has obtained lower bounds for L^∞ norms of eigenfunctions.

Acknowledgments. I am grateful to Greg Martin for a valuable suggestion.

2. LARGE VALUES OF $|\zeta(\frac{1}{2} + it)|$: PROOF OF THEOREM 1

Let Φ be a smooth function compactly supported in $[1, 2]$, such that $0 \leq \Phi(t) \leq 1$ always and $\Phi(t) = 1$ for $t \in (5/4, 7/4)$. Let $\hat{\Phi}(y) = \int_{-\infty}^{\infty} \Phi(t) e^{-ity} dt$ denote the Fourier transform of Φ . Integrating by parts we note that $\hat{\Phi}(y) \ll_\nu |y|^{-\nu}$ for any integer $\nu \geq 1$.

We first show how to evaluate the moments $M_1(R, T)$ and $M_2(R, T)$ defined in (1) when $N \leq T^{1-\epsilon}$. Observe that

$$\begin{aligned} \int_{-\infty}^{\infty} |R(t)|^2 \Phi\left(\frac{t}{T}\right) dt &= \sum_{m, n \leq N} r(m) \overline{r(n)} \int_{-\infty}^{\infty} \left(\frac{n}{m}\right)^{it} \Phi\left(\frac{t}{T}\right) dt \\ &= T \sum_{m, n \leq N} r(m) \overline{r(n)} \hat{\Phi}(T \log(m/n)). \end{aligned}$$

Since $N \leq T^{1-\epsilon}$ we see that if $m \neq n$ then $T |\log(m/n)| \gg T^\epsilon$ so that $\hat{\Phi}(T \log(m/n)) \ll_\epsilon T^{-2}$ say. Therefore

$$\begin{aligned} M_1(R, T) &= T \hat{\Phi}(0) \sum_{n \leq N} |r(n)|^2 + O\left(T^{-1} \left(\sum_{n \leq N} |r(n)|\right)^2\right) \\ (2) \quad &= T \hat{\Phi}(0) (1 + O(T^{-1})) \sum_{n \leq N} |r(n)|^2, \end{aligned}$$

by a simple application of Cauchy's inequality.

Now consider

$$\int_{-\infty}^{\infty} |R(t)|^2 \sum_{k \leq T} \frac{1}{k^{\frac{1}{2} + it}} \Phi\left(\frac{t}{T}\right) dt = T \sum_{m, n \leq N} \sum_{k \leq T} \frac{r(m) \overline{r(n)}}{\sqrt{k}} \hat{\Phi}(T \log(mk/n)).$$

If $N \leq T^{1-\epsilon}$ then for off-diagonal terms $mk \neq n$ we have $\hat{\Phi}(T \log(mk/n)) \ll_\epsilon T^{-2}$. Thus the above equals

$$\begin{aligned} &\hat{\Phi}(0) T \sum_{mk=n \leq N} \frac{r(m) \overline{r(n)}}{\sqrt{k}} + O\left(T^{-1} \sum_{k \leq T} \frac{1}{\sqrt{k}} \left(\sum_{n \leq N} |r(n)|\right)^2\right) \\ &= T \hat{\Phi}(0) \sum_{mk=n \leq N} \frac{r(m) \overline{r(n)}}{\sqrt{k}} + O\left(T^{\frac{1}{2}} \sum_{n \leq N} |r(n)|^2\right). \end{aligned}$$

Since $\zeta(\frac{1}{2} + it) = \sum_{k \leq T} k^{-\frac{1}{2} - it} + O(T^{-\frac{1}{2}})$ for $T \leq t \leq 2T$ (see Theorem 4.11 of [15]) we deduce that

$$(3) \quad M_2(R, T) = T \hat{\Phi}(0) \sum_{mk=n \leq N} \frac{r(m) \overline{r(n)}}{\sqrt{k}} + O\left(T^{\frac{1}{2}} \sum_{n \leq N} |r(n)|^2\right).$$

From (2) and (3) we glean that, if $N \leq T^{1-\epsilon}$ then

$$(4) \quad \max_{T \leq t \leq 2T} |\zeta(\frac{1}{2} + it)| \geq (1 + O(T^{-1})) \left| \sum_{mk \leq N} \frac{r(m) \overline{r(mk)}}{\sqrt{k}} \right| / \left(\sum_{n \leq N} |r(n)|^2 \right) + O(T^{-\frac{1}{2}}).$$

It remains to choose the resonator coefficients $r(n)$ so as to maximize this ratio.

Theorem 2.1. *For large N we have*

$$\max_r \left| \sum_{mk \leq N} \frac{r(m) \overline{r(mk)}}{\sqrt{k}} \right| / \left(\sum_{n \leq N} |r(n)|^2 \right) = \exp \left(\frac{\sqrt{\log N}}{\sqrt{\log \log N}} + O\left(\frac{\sqrt{\log N}}{\log \log N} \right) \right).$$

Proof of the lower bound of Theorem 2.1. We take $r(n)$ to be $f(n)$ where f is a multiplicative function such that $f(p^k) = 0$ for $k \geq 2$. Let $L := \sqrt{\log N \log \log N}$, and define $f(p) = L/(\sqrt{p} \log p)$ if $L^2 \leq p \leq \exp((\log L)^2)$, and $f(p) = 0$ for all other primes p . Note that the denominator in our ratio is

$$(5) \quad \sum_{n \leq N} f(n)^2 \leq \sum_{n=1}^{\infty} f(n)^2 = \prod_p (1 + f(p)^2).$$

Now we need a lower bound for the numerator of our ratio. Below we make use of the observation that if a_n is a sequence of non-negative real numbers then for any $\alpha > 0$ we have

$$\sum_{n > x} a_n \leq x^{-\alpha} \sum_{n > x} a_n n^{\alpha} \leq x^{-\alpha} \sum_{n=1}^{\infty} a_n n^{\alpha}.$$

This observation is often called ‘Rankin’s trick.’ Thus the numerator of our ratio is, for any $\alpha > 0$,

$$\sum_{k \leq N} \frac{f(k)}{\sqrt{k}} \sum_{\substack{n \leq N/k \\ (n,k)=1}} f(n)^2 = \sum_{k \leq N} \frac{f(k)}{\sqrt{k}} \left(\prod_{p \nmid k} (1 + f(p)^2) + O\left(\left(\frac{k}{N} \right)^{\alpha} \prod_{p \nmid k} (1 + p^{\alpha} f(p)^2) \right) \right).$$

The error term above is plainly

$$O\left(\frac{1}{N^{\alpha}} \prod_p (1 + p^{\alpha} f(p)^2 + f(p) p^{\alpha - \frac{1}{2}}) \right),$$

while the main term is

$$\sum_{k \leq N} \frac{f(k)}{\sqrt{k}} \prod_{p \nmid k} (1 + f(p)^2) = \prod_p (1 + f(p)^2 + f(p)/\sqrt{p}) + O\left(\frac{1}{N^\alpha} \prod_p (1 + f(p)^2 + f(p)p^\alpha/\sqrt{p})\right).$$

Thus our numerator is

$$(6) \quad \prod_p (1 + f(p)^2 + f(p)/\sqrt{p}) + O\left(\frac{1}{N^\alpha} \prod_p (1 + p^\alpha f(p)^2 + f(p)p^{\alpha-\frac{1}{2}})\right).$$

Taking $\alpha = 1/(\log L)^3$ we may see that the ratio of the error term in (6) to the main term there is

$$\ll \exp\left(-\alpha \log N + \sum_{L^2 \leq p \leq \exp(\log^2 L)} (p^\alpha - 1) \left(\frac{L}{p \log p} + \frac{L^2}{p \log^2 p}\right)\right) \ll \exp\left(-\alpha \frac{\log N}{\log \log N}\right),$$

with a little calculation using the prime number theorem. Thus for large N the numerator of our ratio is at least

$$(7) \quad \frac{1}{2} \prod_p \left(1 + f(p)^2 + \frac{f(p)}{\sqrt{p}}\right),$$

and the lower bound of the Theorem follows from (5).

Proof of the upper bound of Theorem 2.1. Define the multiplicative function g by setting $g(p^k) = \min(1, L/(p^{k/2} \log p))$ where $L = \sqrt{\log N \log \log N}$ as above. Since $2|r(mk)r(m)| \leq |r(mk)|^2/g(k) + g(k)|r(m)|^2$ we obtain that the numerator of our ratio is

$$\leq \frac{1}{2} \sum_{km \leq N} \frac{1}{\sqrt{k}} \left(\frac{|r(mk)|^2}{g(k)} + g(k)|r(m)|^2 \right) = \frac{1}{2} \sum_{n \leq N} |r(n)|^2 \left(\sum_{k \leq N/n} \frac{g(k)}{\sqrt{k}} + \sum_{k|n} \frac{1}{\sqrt{k}g(k)} \right),$$

with a little regrouping. Note that

$$\begin{aligned} \sum_{k \leq N/n} \frac{g(k)}{\sqrt{k}} &\leq \prod_p \left(1 + \frac{g(p)}{\sqrt{p}-1}\right) \\ &\ll \exp\left(\sum_{p \leq \log N / \log \log N} \frac{1}{\sqrt{p}-1} + \sum_{p > \log N / \log \log N} \frac{L}{\sqrt{p}(\sqrt{p}-1) \log p}\right) \\ &= \exp\left(\frac{\sqrt{\log N}}{\sqrt{\log \log N}} + O\left(\frac{\sqrt{\log N} \log \log \log N}{(\log \log N)^{\frac{3}{2}}}\right)\right). \end{aligned}$$

Further observe that for $n \leq N$

$$\sum_{k|n} \frac{1}{\sqrt{k}g(k)} \leq \prod_{p^a \parallel n} \left(1 + \frac{a \log p}{L}\right) \prod_{p|n} \left(1 + \frac{1}{\sqrt{p}-1}\right).$$

The first factor above is $\leq \exp(\sum_{p^a \parallel n} (a \log p)/L) = n^{1/L} \leq N^{1/L}$. The second factor is $\ll \exp(O(\sqrt{\log N}/\log \log N))$ by a simple calculation using the prime number theorem. The upper bound implicit in the Theorem follows.

Proof of Theorem 1. Using Theorem 2.1 in (4), and choosing $N = T^{1-\epsilon}$ we obtain immediately the first assertion of Theorem 1. It remains now to establish the lower bound on the frequency with which large values are attained. We have

$$T \log T \sim \int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^2 dt \leq \frac{1}{2} T \log T + \int_{\substack{t \in [T, 2T] \\ |\zeta(\frac{1}{2} + it)| \geq \sqrt{\frac{1}{2} \log T}}} |\zeta(\tfrac{1}{2} + it)|^2 dt,$$

so that

$$\begin{aligned} T \log T &\ll \int_{\substack{t \in [T, 2T] \\ |\zeta(\frac{1}{2} + it)| \geq \sqrt{\frac{1}{2} \log T}}} |\zeta(\tfrac{1}{2} + it)|^2 dt \\ &\ll \left(\text{meas}\{t \in [T, 2T] : |\zeta(\tfrac{1}{2} + it)| \geq \sqrt{\tfrac{1}{2} \log T}\} \right)^{\frac{1}{2}} \left(\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^4 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Since $\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^4 dt \asymp T(\log T)^4$ we conclude that

$$\text{meas}\{t \in [T, 2T] : |\zeta(\tfrac{1}{2} + it)| \geq \sqrt{\tfrac{1}{2} \log T}\} \gg \frac{T}{(\log T)^2},$$

which gives our desired lower bound when $3 \leq V \leq \frac{1}{2} \log(\frac{1}{2} \log T)$.

For larger values of V , we use the resonator method with $N = T^{\frac{1}{2}-\epsilon}$. If $2e^V M_1(R, T) \leq |M_2(R, T)|$ (with the resonator R still to be chosen) then

$$|M_2(R, T)| \leq e^V M_1(R, T) + \int_{\{t: |\zeta(\frac{1}{2} + it)| \geq e^V\}} |\zeta(\tfrac{1}{2} + it)| |R(t)|^2 \Phi(\tfrac{t}{T}) dt.$$

Using Cauchy's inequality twice, we see that the integral above is

$$\leq \left(\text{meas}\{t \in [T, 2T] : |\zeta(\tfrac{1}{2} + it)| \geq e^V\} \right)^{\frac{1}{4}} \left(\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^4 dt \right)^{\frac{1}{4}} \left(\int_{-\infty}^{\infty} |R(t)|^4 \Phi(\tfrac{t}{T}) dt \right)^{\frac{1}{2}}.$$

Therefore

$$(8) \quad \text{meas}\{t \in [T, 2T] : |\zeta(\tfrac{1}{2} + it)| \geq e^V\} \gg \frac{|M_2(R, T)|^4}{T \log^4 T} \left(\int_{-\infty}^{\infty} |R(t)|^4 \Phi(\tfrac{t}{T}) dt \right)^{-2}.$$

Let A be large with $10A^2 \log A \leq \log N$. We choose the resonator coefficients $r(n)$ to be multiplicative, with $r(p^k) = 0$ for $k \geq 2$ and

$$r(p) = \begin{cases} A/\sqrt{p} & \text{if } A^2 \leq p \leq N^{\frac{1}{2A^2}} \\ 0 & \text{otherwise.} \end{cases}$$

We use Rankin's trick and argue as in the proof of the lower bound of Theorem 2.1 (taking now $\alpha = A^2/\log N$). That gives

$$(9) \quad \frac{|M_2(R, T)|}{M_1(R, T)} \geq \frac{1}{2} \prod_p \left(1 + r(p)^2 + \frac{r(p)}{\sqrt{p}} \right) (1 + r(p)^2)^{-1} = \exp \left((1 + o(1)) A \log \frac{\log N}{4A^2 \log A} \right).$$

Further,

$$\int_{-\infty}^{\infty} |R(t)|^4 \Phi\left(\frac{t}{T}\right) dt = \sum_{a,b,c,d \leq N} r(a)r(b)r(c)r(d) T \hat{\Phi}(T \log \frac{ab}{cd}).$$

If $N \leq T^{\frac{1}{2}-\epsilon}$ then if $ab \neq cd$ then $|\log \frac{ab}{cd}| \gg T^{-1+\epsilon}$ so that $\hat{\Phi}(T \log \frac{ab}{cd}) \ll_{\epsilon} T^{-4}$, say. Since $r(n) \leq 1$ for all n we conclude that the off-diagonal terms $ab \neq cd$ contribute an amount $\ll T^{-3}N^4 \ll T^{-1}$. Thus

$$\begin{aligned} \int_{-\infty}^{\infty} |R(t)|^4 \Phi\left(\frac{t}{T}\right) dt &= T \hat{\Phi}(0) \sum_{\substack{a,b,c,d \leq N \\ ab=cd}} r(a)r(b)r(c)r(d) + O(T^{-1}) \\ &\ll T \prod_p (1 + 4r(p)^2 + r(p)^4). \end{aligned}$$

Since $M_2(R, T) \gg T$, we conclude from (8) that

$$\begin{aligned} \text{meas}\{t \in [T, 2T] : |\zeta(\tfrac{1}{2} + it)| \geq e^V\} &\gg \frac{T}{\log^4 T} \exp \left(-4 \sum_p r(p)^2 \right) \\ &\gg \frac{T}{\log^4 T} \exp \left(-5A^2 \log \frac{\log N}{4A^2 \log A} \right). \end{aligned}$$

We may choose $A \sim V(\log \frac{\log N}{4V^2 \log V})^{-1}$ such that the RHS of (9) exceeds $2e^V$, and then the above estimate yields the bound claimed in Theorem 1.

3. EXTREME VALUES OF QUADRATIC DIRICHLET L-FUNCTIONS: PROOF OF THEOREM 2

For convenience, we restrict ourselves to fundamental discriminants of the form $8d$ where d is an odd, squarefree number with $X/16 \leq d \leq X/8$. As before, we will consider the two moments

$$M_1(R, X) = \sum_{X/16 \leq d \leq X/8} \mu(2d)^2 R(8d)^2, \quad M_2(R, X) = \sum_{X/16 \leq d \leq X/8} \mu(2d)^2 L(\tfrac{1}{2}, \chi_{8d}) R(8d)^2,$$

where

$$R(8d) = \sum_{n \leq N} r(n) \left(\frac{8d}{n} \right),$$

is a resonator, whose coefficients $r(n)$ are real numbers to be chosen presently.

Lemma 3.1. *The quantity $M_1(R, X)$ equals*

$$\frac{X}{16\zeta(2)} \sum_{\substack{n_1, n_2 \leq N \\ n_1 n_2 = \text{odd square}}} r(n_1) r(n_2) \prod_{p|2n_1 n_2} \left(\frac{p}{p+1} \right) + O\left(X^{\frac{1}{2}+\epsilon} N^{\frac{1}{2}} \left(\sum_{n \leq N} |r(n)| \right)^2 \right).$$

Proof. Expanding $R(8d)^2$ we see that

$$(10) \quad M_1(R, X) = \sum_{n_1, n_2 \leq N} r(n_1) r(n_2) \sum_{X/16 \leq d \leq X/8} \mu(2d)^2 \left(\frac{8d}{n_1 n_2} \right).$$

Let n be an odd number and $z \geq 3$. We record the following character sum estimate which may be obtained easily from the Pólya-Vinogradov inequality (or see Lemma 3.1 of [12] for details). If n is not a perfect square then

$$(11a) \quad \sum_{d \leq z} \mu(2d)^2 \left(\frac{8d}{n} \right) \ll z^{\frac{1}{2}} n^{\frac{1}{4}} \log(2n),$$

while if n is a perfect square then

$$(11b) \quad \sum_{d \leq z} \mu(2d)^2 \left(\frac{8d}{n} \right) = \frac{z}{\zeta(2)} \prod_{p|2n} \left(\frac{p}{p+1} \right) + O(z^{\frac{1}{2}+\epsilon} n^{\epsilon}).$$

The Lemma follows upon using (11a,b) in (10).

To evaluate $M_2(R, X)$ we will use (11a,b) along with a standard “approximate functional equation.” The approximate functional equation we need states that for an odd, positive, square-free number d we have

$$L\left(\frac{1}{2}, \chi_{8d}\right) = 2 \sum_{n=1}^{\infty} \frac{\chi_{8d}(n)}{\sqrt{n}} W\left(\frac{n\sqrt{\pi}}{\sqrt{8d}}\right)$$

where the weight W is defined by

$$W(\xi) = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(\frac{s}{2} + \frac{1}{4})}{\Gamma(\frac{1}{4})} \xi^{-s} \frac{ds}{s},$$

and the integral is over a vertical line $c - i\infty$ to $c + i\infty$ with $c > 0$. The weight $W(\xi)$ is smooth and satisfies $W(\xi) = 1 + O(\xi^{\frac{1}{2}-\epsilon})$ for small ξ , and $W(\xi) \ll e^{-\xi}$ for large ξ . Moreover the derivative $W'(\xi)$ satisfies $W'(\xi) \ll \xi^{\frac{1}{2}-\epsilon} e^{-\xi}$. These facts are easily established; for details see Lemmas 2.1 and 2.2 of [13], or Lemma 3.2 of [12].

Lemma 3.2. *The quantity $M_2(R, X)$ equals*

$$\begin{aligned} & \frac{X}{8\zeta(2)} \sum_{n_1, n_2 \leq N} r(n_1)r(n_2) \sum_{\substack{n \\ nn_1n_2 = \text{odd square}}} \frac{1}{\sqrt{n}} \prod_{p|2nn_1n_2} \left(\frac{p}{p+1} \right) \int_1^2 W\left(\frac{n\sqrt{2\pi}}{\sqrt{Xt}} \right) dt \\ & + O\left(X^{\frac{7}{8}+\epsilon} N^{\frac{1}{2}} \left(\sum_{n \leq N} |r(n)| \right)^2 \right). \end{aligned}$$

Proof. Expanding $R(8d)^2$, and using the approximate functional equation, we have that

$$M_2(R, X) = 2 \sum_{n_1, n_2 \leq N} r(n_1)r(n_2) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sum_{X/16 \leq d \leq X/8} \mu(2d)^2 \left(\frac{8d}{nn_1n_2} \right) W\left(\frac{n\sqrt{\pi}}{\sqrt{8d}} \right).$$

By (11a,b) and partial summation we see that if nn_1n_2 is not an odd square then

$$\sum_{X/16 \leq d \leq X/8} \mu(2d)^2 \left(\frac{8d}{nn_1n_2} \right) W\left(\frac{n\sqrt{\pi}}{\sqrt{8d}} \right) \ll X^{\frac{1}{2}} (nn_1n_2)^{\frac{1}{4}+\epsilon} e^{-n/\sqrt{X}},$$

while if nn_1n_2 is an odd square that sum over d is

$$\frac{X}{16\zeta(2)} \prod_{p|2nn_1n_2} \left(\frac{p}{p+1} \right) \int_1^2 W\left(\frac{n\sqrt{2\pi}}{\sqrt{Xt}} \right) dt + O(X^{\frac{1}{2}+\epsilon} e^{-n/\sqrt{X}}).$$

The errors above contribute to $M_2(R, X)$ an amount

$$\ll X^{\frac{1}{2}+\epsilon} N^{\frac{1}{2}} \left(\sum_{\ell \leq N} |r(\ell)| \right)^2 \sum_{n=1}^{\infty} \frac{n^{\frac{1}{4}+\epsilon}}{\sqrt{n}} e^{-n/\sqrt{X}} \ll X^{\frac{7}{8}+\epsilon} N^{\frac{1}{2}} \left(\sum_{n \leq N} |r(n)| \right)^2.$$

The Lemma follows.

Proposition 3.3. *Let $N \leq X^{\frac{1}{20}-\epsilon}$ be large. Set $L = \sqrt{\log N \log \log N}$ and choose the resonator coefficients $r(n)$ to be $\mu(n)f(n)$ where f is a multiplicative function with $f(p) = L/(\sqrt{p} \log p)$ for $L^2 \leq p \leq \exp((\log L)^2)$ and $f(p) = 0$ for all other primes.² Then*

$$M_1(R, X) \sim \frac{X}{24\zeta(2)} \prod_p (1 + f(p)^2),$$

and

$$M_2(R, X) \sim C_1 X (\log X) \prod_p \left(1 + f(p)^2 - 2 \frac{f(p)}{\sqrt{p}} \right),$$

where C_1 is an absolute positive constant.

²Thus f is the function appearing in the proof of the lower bound in Theorem 2.1.

Taking $N = X^{\frac{1}{20}-\epsilon}$, a little calculation shows that

$$\frac{M_2(R, X)}{M_1(R, X)} = \exp\left(- (2 + o(1)) \frac{\sqrt{\log N}}{\sqrt{\log \log N}}\right) = \exp\left(- \left(\frac{1}{\sqrt{5}} + o(1)\right) \frac{\sqrt{\log X}}{\sqrt{\log \log X}}\right).$$

This demonstrates the existence of the small values claimed in Theorem 2. To find large values we take $r(n) = f(n)$ in Proposition 3.3, and argue in an identical manner.

Proof of Proposition 3.3. Lemma 3.1 gives

$$M_1(R, X) = \frac{X}{16\zeta(2)} \sum_{n \leq N} \mu(n)^2 f(n)^2 \prod_{p|2n} \left(\frac{p}{p+1}\right) + O(X^{\frac{1}{2}+\epsilon} N^{\frac{5}{2}}).$$

Rankin's trick shows that for any $\alpha > 0$

$$\sum_{n \leq N} \mu(n)^2 f(n)^2 \prod_{p|2n} \left(\frac{p}{p+1}\right) = \frac{2}{3} \prod_p \left(1 + f(p)^2 \frac{p}{p+1}\right) + O\left(N^{-\alpha} \prod_p \left(1 + f(p)^2 p^\alpha \frac{p}{p+1}\right)\right).$$

Choosing $\alpha = 1/(\log L)^3$ (as in Theorem 2.1) we find that the ratio of the error term above to the main term is $\ll \exp(-\alpha \log N / \log \log N)$. When $N \leq X^{\frac{1}{5}-\epsilon}$ we conclude that

$$M_1(R, X) \sim \frac{X}{24\zeta(2)} \prod_p \left(1 + f(p)^2 \frac{p}{p+1}\right) \sim \frac{X}{24\zeta(2)} \prod_p (1 + f(p)^2).$$

This proves the first assertion of the Proposition.

Now we turn to $M_2(R, X)$. We use Lemma 3.2, and note that when $N \leq X^{\frac{1}{20}-\epsilon}$ the remainder term there is $O(X^{1-\epsilon})$. Consider the main term in the asymptotic formula of Lemma 3.2. To analyze this we write $n_1 = ar$ and $n_2 = as$ where $a = (n_1, n_2)$ so that $(r, s) = 1$; from our choice of the coefficients $r(n)$, we also have that $(a, r) = (a, s) = 1$. With this notation, we may write the variable n in Lemma 3.2 as $rs m^2$ for some odd integer m . Thus the main term in Lemma 3.2 equals

$$(12) \quad \frac{X}{12\zeta(2)} \sum_{\substack{a, r, s \\ ar, as \leq N \\ (a, r) = (a, s) = (r, s) = 1}} \mu(a)^2 f(a)^2 \frac{\mu(r) f(r) \mu(s) f(s)}{\sqrt{rs}} \times \sum_{m \text{ odd}} \frac{1}{m} \prod_{p|arsm} \left(\frac{p}{p+1}\right) \int_1^2 W\left(\frac{rsm^2 \sqrt{2\pi}}{\sqrt{X}t}\right) dt.$$

We now evaluate the sum over m above. Recalling the definition of $W(\xi)$ we may express that sum as

$$(13) \quad \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(\frac{w}{2} + \frac{1}{4})}{\Gamma(\frac{1}{4})} \left(\frac{X}{rs\sqrt{2\pi}}\right)^{\frac{w}{2}} \left(\int_1^2 t^{\frac{w}{2}} dt\right) \sum_{m \text{ odd}} \frac{1}{m^{1+w}} \prod_{p|arsm} \left(\frac{p}{p+1}\right) \frac{dw}{w},$$

where the integral is over the line from $c - i\infty$ to $c + i\infty$ with $c > 0$. A little calculation allows us to write the sum over m above as

$$\zeta(1+w)(1-2^{-(1+w)}) \prod_{p|ars} \left(\frac{p}{p+1}\right) \prod_{p \nmid 2ars} \left(1 - \frac{1}{p^{1+w}(p+1)}\right).$$

We insert this in (13) and move the line of integration to $\operatorname{Re} w = -\frac{1}{2} + \epsilon$. In view of the rapid decay of $\Gamma(\frac{w}{2} + \frac{1}{4})$, the integral on that line is $\ll (X/rs)^{-\frac{1}{4}+\epsilon}$, and therefore (13) equals

$$\begin{aligned} \prod_{p|ars} \left(\frac{p}{p+1}\right) \operatorname{Res}_{w=0} \frac{\Gamma(\frac{w}{2} + \frac{1}{4})}{\Gamma(\frac{1}{4})} \left(\frac{X}{rs\sqrt{2\pi}}\right)^{\frac{w}{2}} \left(\int_1^2 t^{\frac{w}{2}} dt\right) \frac{\zeta(1+w)}{w} (1-2^{-1-w}) \\ \times \prod_{p \nmid 2ars} \left(1 - \frac{1}{p^{1+w}(p+1)}\right) + O(X^{-\frac{1}{4}+\epsilon}(rs)^{\frac{1}{4}}). \end{aligned}$$

Computing the residue, we see that the above equals

$$\frac{1}{2} \prod_{p|ars} \left(\frac{p}{p+1}\right) \prod_{p \nmid 2ars} \left(1 - \frac{1}{p(p+1)}\right) \left(\log \frac{X}{rs} + C - \sum_{p|ars} \frac{\log p}{p(p+1)}\right) + O(X^{-\frac{1}{4}+\epsilon}(rs)^{\frac{1}{4}}),$$

for a suitable absolute constant C . Using this in (12) we conclude that for $N \leq X^{\frac{1}{20}-\epsilon}$

$$\begin{aligned} M_2(R, X) = C_1 X \sum_{\substack{a, r, s \\ ar, as \leq N \\ (a, r) = (a, s) = (r, s) = 1}} \mu(a)^2 f(a)^2 h(a) \frac{\mu(r)f(r)h(r)}{\sqrt{r}} \frac{\mu(s)f(s)h(s)}{\sqrt{s}} \\ (14) \quad \times \left(\log \frac{X}{rs} + C - \sum_{p|ars} \frac{\log p}{p(p+1)}\right) + O(X^{1-\epsilon}), \end{aligned}$$

where C_1 is an absolute positive constant, and h is a completely multiplicative function defined by $h(p) = p^2/(p^2 + p - 1)$.

To simplify (14) further, we first extend the summations over a , r , and s to run over all integers, and then use Rankin's trick to estimate the tails. The extended sum equals

$$\begin{aligned} \sum_{\substack{a, r, s \\ (a, r) = (a, s) = (r, s) = 1}} \mu(a)^2 f(a)^2 h(a) \frac{\mu(r)f(r)h(r)}{\sqrt{r}} \frac{\mu(s)f(s)h(s)}{\sqrt{s}} \\ \times \left(\log X + C - \sum_{p|rs} \log p - \sum_{p|ars} \frac{\log p}{p(p+1)}\right). \end{aligned}$$

By multiplicativity this is seen to be

$$\begin{aligned} \prod_p \left(1 + f(p)^2 h(p) - 2 \frac{f(p)h(p)}{\sqrt{p}}\right) \\ \times \left(\log X + C + \sum_{\ell \text{ prime}} \log \ell \frac{2f(\ell)h(\ell)(1 + 1/(\ell(\ell+1)))/\sqrt{\ell} - f(\ell)^2 h(\ell)/(\ell(\ell+1))}{1 + f(\ell)^2 h(\ell) - 2f(\ell)h(\ell)/\sqrt{\ell}}\right). \end{aligned}$$

Since $h(p) = 1 + O(1/p)$ we may further simplify the above to

$$(15) \quad \sim (\log X) \prod_p \left(1 + f(p)^2 - 2 \frac{f(p)}{\sqrt{p}} \right).$$

It remains to bound the error incurred upon extending the sums to infinity. By symmetry we may suppose that $ar > N$, and we wish to estimate

$$\sum_{\substack{a,r,s \\ ar > N}} f(a)^2 \frac{f(r)f(s)}{\sqrt{rs}} (\log X + \log r) \ll \prod_p \left(1 + \frac{f(p)}{\sqrt{p}} \right) \sum_{ar > N} f(a)^2 \frac{f(r)}{\sqrt{r}} (\log X + \log r).$$

As before we will use Rankin's trick with $\alpha = 1/(\log L)^3$. If $ar > N$ then we have $(\log X + \log r) \ll (\log X) N^{-\alpha} (ar)^\alpha$. Therefore, our desired quantity is

$$\begin{aligned} &\ll (\log X) \prod_p \left(1 + \frac{f(p)}{\sqrt{p}} \right) N^{-\alpha} \sum_{a,r} f(a)^2 a^\alpha \frac{f(r) r^\alpha}{\sqrt{r}} \\ &\ll (\log X) N^{-\alpha} \prod_p \left(1 + \frac{f(p)}{\sqrt{p}} \right) \left(1 + f(p)^2 p^\alpha + \frac{f(p) p^\alpha}{\sqrt{p}} \right). \end{aligned}$$

The ratio of the above quantity to that in (15) is

$$\ll \exp \left(-\alpha \log N + \sum_p \left(f(p)^2 (p^\alpha - 1) + 4f(p) p^{\alpha - \frac{1}{2}} \right) \right) \ll \exp \left(-\alpha \frac{\log N}{\log \log N} \right).$$

Combining this estimate with (14) and (15) we conclude that

$$M_2(R, X) \sim C_1 X (\log X) \prod_p \left(1 + f(p)^2 - 2 \frac{f(p)}{\sqrt{p}} \right).$$

This completes the proof of the Proposition.

4. EXTREME VALUES OF L -FUNCTIONS OF CUSP FORMS: PROOF OF THEOREM 3

Given $f \in H_k$ we define

$$(16a) \quad \omega(f) := \frac{3}{\pi} \frac{(4\pi)^k}{\Gamma(k)} \|f\|^2,$$

where $\|f\|^2 = \langle f, f \rangle = \int_{\Gamma \backslash \mathbb{H}} y^k |f(z)|^2 \frac{dx dy}{y^2}$ is the Petersson norm of f . The weights $\omega(f)$ are related to the value at 1 of the symmetric square L -function of f . Namely, (see Iwaniec [7] for example)

$$(16b) \quad \omega(f) = L(1, \text{sym}^2 f) / \zeta(2).$$

We also know that the weights $\omega(f)$ are roughly of constant size; precisely,

$$(16c) \quad (\log k)^{-2} \ll \omega(f) \ll (\log k)^2.$$

For any two integers $m, n \geq 1$ we have

$$(17) \quad \frac{12}{k-1} \sum_{f \in H_k} \frac{\lambda_f(m) \lambda_f(n)}{\omega(f)} = \delta_{m,n} + 2\pi i^k \sum_{c=1}^{\infty} \frac{S(m, n; c)}{c} J_{k-1} \left(\frac{4\pi \sqrt{mn}}{c} \right),$$

where $\delta_{m,n} = 1$ or 0 depending on whether $m = n$ or not, J_{k-1} is the usual Bessel function, and $S(m, n; c) = \sum_{a \pmod{c}}^* e\left(\frac{am + \bar{a}n}{c}\right)$ is Kloosterman's sum. This is Petersson's formula, see Iwaniec [7].

If $x \leq 2k$ then

$$|J_{k-1}(x)| = \left| \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!(\ell+k-1)!} \left(\frac{x}{2}\right)^{2\ell+k-1} \right| \leq \frac{(x/2)^{k-1}}{(k-1)!} \sum_{\ell=0}^{\infty} \frac{(x/2)^{2\ell}}{\ell! k^\ell} \leq \frac{e^{x/2} (x/2)^{k-1}}{(k-1)!}.$$

Using this together with $|S(m, n; c)| \leq c$ we obtain that if $4\pi\sqrt{mn} \leq k/10$ then

$$(18) \quad \begin{aligned} \frac{12}{k-1} \sum_{f \in H_k} \frac{\lambda_f(m) \lambda_f(n)}{\omega(f)} &= \delta_{m,n} + O\left(\frac{e^{2\pi\sqrt{mn}}}{(k-1)!} \sum_{c=1}^{\infty} \left(\frac{2\pi\sqrt{mn}}{c}\right)^{k-1}\right) \\ &= \delta_{m,n} + O(e^{-k}). \end{aligned}$$

Let $r(n)$ be arbitrary real numbers and consider the resonator $R(f) = \sum_{n \leq N} \lambda_f(n) r(n)$. If $N \leq k/(40\pi)$ then we obtain from (18) that

$$(19) \quad \frac{12}{k-1} \sum_{f \in H_k} \frac{R(f)^2}{\omega(f)} = \sum_{m, n \leq N} r(m) r(n) (\delta_{m,n} + O(e^{-k})) = \sum_{n \leq N} r(n)^2 (1 + O(ke^{-k})),$$

where the last equality follows from Cauchy's inequality.

Next we want to calculate the weighted average of $|R(f)|^2 L(\frac{1}{2}, f)$. To do this we require an “approximate functional equation” for $L(\frac{1}{2}, f)$ which we now describe briefly. We consider, for some $c > \frac{1}{2}$,

$$(20a) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (2\pi)^{-s} \frac{\Gamma(s + \frac{k}{2})}{\Gamma(\frac{k}{2})} L(s + \frac{1}{2}, f) \frac{ds}{s}.$$

We move the line of integration to the line $\operatorname{Re}(s) = -c$ and use the functional equation. The pole at $s = 0$ leaves the residue $L(\frac{1}{2}, f)$ and thus (20a) equals

$$L(\frac{1}{2}, f) + \frac{i^k}{2\pi i} \int_{-c-i\infty}^{-c+i\infty} (2\pi)^{-s} \frac{\Gamma(-s + \frac{k}{2})}{\Gamma(\frac{k}{2})} L(-s + \frac{1}{2}, f) \frac{ds}{s}.$$

Replacing $-s$ by s we deduce that

$$(20b) \quad L(\tfrac{1}{2}, f) = (1 + i^k) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (2\pi)^{-s} \frac{\Gamma(s + \frac{k}{2})}{\Gamma(\frac{k}{2})} L(s + \tfrac{1}{2}, f) \frac{ds}{s}.$$

Defining, for real numbers $x > 0$,

$$(20c) \quad V(x) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (2\pi)^{-s} \frac{\Gamma(s + \frac{k}{2})}{\Gamma(\frac{k}{2})} x^{-s} \frac{ds}{s},$$

and expanding $L(s + \frac{1}{2}, f)$ into its Dirichlet series we deduce from (20b) that

$$(20d) \quad L(\tfrac{1}{2}, f) = (1 + i^k) \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{\sqrt{n}} V(n).$$

Moving the line of integration in (20c) to $c = k/2$ and $c = 1 - k/2$ we obtain respectively that

$$(20e) \quad V(x) \ll \left(\frac{k}{2\pi x}\right)^{\frac{k}{2}}, \quad \text{and} \quad V(x) = 1 + O\left(\frac{(2\pi x)^{\frac{k}{2}-1}}{\Gamma(\frac{k}{2})}\right).$$

Suppose now that $N \leq \sqrt{k}/100$. Then, using the Hecke relations and (18), we obtain that

$$\begin{aligned} & \frac{12}{k-1} \sum_{f \in H_k} \frac{R(f)^2}{\omega(f)} \sum_{r \leq 2k} \frac{\lambda_f(r)}{\sqrt{r}} V(r) \\ &= \frac{12}{k-1} \sum_{f \in H_k} \frac{1}{\omega(f)} \sum_{m, n \leq N} r(m)r(n) \sum_{d|(m, n)} \lambda_f\left(\frac{mn}{d^2}\right) \sum_{r \leq 2k} \frac{\lambda_f(r)}{\sqrt{r}} V(r) \\ &= \sum_{m, n \leq N} r(m)r(n) \sum_{d|(m, n)} \left(\frac{d}{\sqrt{mn}} V\left(\frac{mn}{d^2}\right) + O(ke^{-k}) \right) \\ &= \sum_{m, n \leq N} r(m)r(n) \frac{\sigma((m, n))}{\sqrt{mn}} + O\left(k^3 e^{-k} \sum_{n \leq N} r(n)^2\right). \end{aligned}$$

The final inequality above follows upon using (20e) to replace $V(mn/d^2)$ by 1, and then noting that $\sum_{d|(m, n)} 1 \leq k$ and that $\sum_{m, n \leq N} |r(m)r(n)| \leq M \sum_{n \leq N} r(n)^2$ by Cauchy's inequality. Now suppose that $k \equiv 0 \pmod{4}$. Note that by (20e) the terms $n > 2k$ contribute an amount $O(e^{-k})$ to $L(\frac{1}{2}, f)$. Therefore we deduce that

$$(21) \quad \frac{12}{k-1} \sum_{f \in H_k} \frac{R(f)^2}{\omega(f)} L(\tfrac{1}{2}, f) = \sum_{m, n \leq N} r(m)r(n) \frac{\sigma((m, n))}{\sqrt{mn}} + O\left(k^3 e^{-k} \sum_{n \leq N} r(n)^2\right).$$

To produce large values of $L(\frac{1}{2}, f)$ we choose $N = \sqrt{k}/100$, and choose the resonator coefficients $r(n)$ to be $f(n)$, where f is the multiplicative function used in the proof of the lower bound in Theorem 2.1. Using Rankin's trick in (19) we obtain that

$$\frac{12}{k-1} \sum_{f \in H_k} \frac{R(f)^2}{\omega(f)} \sim \prod_p (1 + f(p)^2).$$

Further, Rankin's trick and (21) give

$$\frac{12}{k-1} \sum_{f \in H_k} \frac{R(f)^2}{\omega(f)} L(\frac{1}{2}, f) \sim \prod_p \left(1 + f(p)^2 \left(1 + \frac{1}{p} \right) + 2 \frac{f(p)}{\sqrt{p}} \right),$$

and the conclusion of Theorem 3 regarding large values follows. To obtain the conclusion concerning small values, we choose $r(n)$ to be $\mu(n)f(n)$.

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